AFFINE ISOMETRIC ACTIONS ON HILBERT SPACES & AMENABILITY

COURSE GIVEN BY ALAIN VALETTE

Contents

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Disclaimer

These are notes from a course given by Alain Valette during the minisemester "Amenability beyond groups" at the Erwin Schrödinger Institute in Vienna in March 2007. The reader is asked to bear in mind the informal nature of course notes.

The letter G will be reserved to denote a group G. This group will most of the time be a topological group and the topology will most of the time be assumed to be locally compact. It will be clear from the context what assumptions we put on G. Similarly π will always denote a representation of G. In case this representation acts on a Hilbert space (usually denoted $\mathcal{H}$ or $\mathcal{H}_\pi$) we will assume that π is unitary and strongly continuous. These and similar conventions will be used throughout these notes without further explanation.

1. Affine actions

1.1. **1-cohomology.** Let G be a group and let V be a vector space (over some field k). An *affine action* of G on V is a homomorphism $\alpha : G \to \mathrm{Aff}(V)$, where $\mathrm{Aff}(V)$ is the group of affine bijections $V \to V$. We have the split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & \mathrm{Aff}(V) \arrow[r, shift left] & \mathrm{GL}(V) \arrow[l, shift left] \arrow[r] & 1 \end{tikzcd}$$

where V is regarded as the group of translations acting on V. Thus α gives a representation $\pi : G \to \mathrm{GL}(V)$ called the *linear part* of α.

Let us ask a converse question: if $\pi : G \to \mathrm{GL}(V)$ is a representation, what are affine actions α with linear part π? Such α must be of the form

$$\alpha(g)v = \pi(g)v + b(g)$$

for any $v \in V$. The vector $b(g)$ is called the *translation part* of α.

From the fact that α is a homomorphism it easily follows that $b : G \to V$ must satisfy the 1-cocycle relation:

$$b(gh) = \pi(g)b(h) + b(g) \tag{1.1}$$

for all $g, h \in G$.

Example 1.1. If π is the trivial representation of G on V then $b \in \mathrm{Hom}(G, V)$, where V is regarded as the additive group.

Notation 1.2. By $Z^1(G, \pi)$ we shall denote the set of all 1-cocycles $G \to V$, i.e. all maps b satisfying (1.1). It is easy to see that $Z^1(G, \pi)$ is a vector space under pointwise operations. The symbol $B^1(G, \pi)$ will denote the set of 1-*coboundaries*, i.e. those $b \in Z^1(G, \pi)$ for which there exists a vector $v \in V$ such that

$$b(g) = \pi(v) - v$$

for all $g \in G$. Clearly $B^1(G, \pi)$ is a subspace of $Z^1(G, \pi)$. Finally we define

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$$

and call $H^1(G, \pi)$ the *first cohomology group of G with coefficients in the G-module V.*

We can now write down a sort of a dictionary between concepts of geometric and algebraic nature:

Affine actions with linear part π	$Z^1(G, \pi)$
Affine actions with linear part π and with a globally fixed point (i.e. conjugate to π via a translation)	$B^1(G, \pi)$
Affine actions up to conjugation by a translation	$H^1(G, \pi)$

Each line of the above table represents a one to one correspondence.

1.2. **Affine isometric actions on Hilbert spaces.** Let $\mathcal{H}$ be a Hilbert space and let $\mathrm{Isom}(\mathcal{H})$ denote the group of affine isometries of $\mathcal{H}$. Let α be a homomorphism $G \to \mathrm{Isom}(\mathcal{H})$.

Remark 1.3. The Mazur-Ulam theorem says that if E is a real Banach space then any isometry of E is affine. For complex Banach spaces we might have to compose with complex conjugation.

For strictly convex Banach spaces (e.g. Hilbert spaces) this is quite easy because we then have a metric characterization of segments: for $x, y \in E$ the segment $[x, y]$ between x and y is

$$\big\{z \in E \big| \|x - z\| + \|z - y\| = \|x - y\|\big\}.$$

In particular any isometry must preserve segments.

For a topological group G we will always assume that affine actions are continuous in the sense that the map

$$G \times \mathcal{H} \ni (g, v) \longmapsto \alpha(g)v \in \mathcal{H}$$

is continuous. The linear part of an isometric affine action is then a strongly continuous unitary representation. We will stick to this setting for the rest of these notes.

Definition 1.4. An affine action α of G on $\mathcal{H}$ *almost has fixed points* if

$$\forall\, \varepsilon > 0 \ \forall\, K \Subset G \ \exists\, v \in \mathcal{H} \ \sup_{g \in K} \big\|\alpha(g)v - v\big\| < \varepsilon.$$

We can endow $Z^1(G, \pi)$ with the topology of uniform convergence on compact subsets and add one more line to the dictionary:

Affine actions which almost have a fixed point	The closure of $B^1(G, \pi)$ in $Z^1(G, \pi)$

We define the *reduced cohomology group* $\overline{H}^2(G, \pi)$ as the quotient

$$\overline{H}^1(G, \pi) = Z^1(G, \pi)/\overline{B^1(G, \pi)}. \tag{1.2}$$

Let us now give a useful characterization of coboundaries. Remember that any 1-cocycle is, in particular, a function $G \to \mathcal{H}$, so we can speak about bounded cocycles.

Proposition 1.5. *Let $b \in Z^1(G, \pi)$. Then*

$$\Big(b \in B^1(G, \pi)\Big) \iff \Big(b \text{ is bounded}\Big).$$

Proof. "$\Rightarrow$" If $b(g) = \pi(g)v - v$ for some fixed $v \in \mathcal{H}$ and all $g \in G$, we have $\big\|b(g)\big\| \le 2\|v\|$.

"$\Leftarrow$" Every bounded set B in a Hilbert space has a circumball (i.e. a closed ball containing B with minimal radius). Thus if B is invariant under some group of isometries then so is its circumball. It follows that the circumcenter (the center of the circumball) also is invariant under the group.[a]

Let α be the affine action associated to b (now assumed to be bounded). Then for any $g \in G$ and $v \in \mathcal{H}$ we have $\alpha(g)v = \pi(g)v + b(g)$. The set $b(G)$ is the orbit of $0 \in \mathcal{H}$ under α. As this set is bounded and α-invariant, the circumcenter of $b(G)$ is α-fixed. Thus $b \in B^1(G, \pi)$. □

1.2.1. *Remarks.* Let us begin with the following theorem:

Theorem 1.6 (Delorme, Guichardet (1973))**.** *Let G be a locally compact group. Then*

1. *If G has property* (T) *then every affine isometric action of G on a Hilbert space has a fixed point. In particular $H^1(G, \pi) = \{0\}$ for any unitary representation π.*
2. *If G is σ-compact then the converse of* (1) *is true.*

It is also known that the converse of (1) of the above theorem is not true without assumption of σ-compactness of G (de Cornulier, 2005).

For the second remark we need the definition:

[a]This statement is called the *lemma of the center.*

Definition 1.7. A locally compact group G has the *Haagerup property* (or is *a-T-menable*) if G admits a metrically proper affine isometric action α on a Hilbert space $\mathcal{H}$, i.e. such that

$$\forall\, v \in \mathcal{H} \ \lim_{g\to\infty} \left\|\alpha(g)v\right\| = +\infty.$$

Let us remark that an affine isometric action is proper if and only if the norm of the associated cocycle is a proper function. Indeed, taking the special case of $v = 0$ in Definition 1.7 we see that $\lim_{g\to\infty} \left\|b(g)\right\| = +\infty$. Therefore $g \mapsto \left\|b(g)\right\|$ is a proper function. Conversely if $g \mapsto \left\|b(g)\right\|$ is proper then for any $v \in \mathcal{H}$

$$\left\|\alpha(g)v\right\| = \left\|\pi(g)v + b(g)\right\| \geq \Big|\left\|b(g)\right\| - \left\|\pi(g)v\right\|\Big| \xrightarrow[g\to\infty]{} +\infty.$$

The class of a-T-menable groups contains σ-compact amenable groups, free groups, Coxeter groups, every closed subgroup of $SO(n,1)$, $SU(n,1)$, etc. ...

Theorem 1.8 (Higson, Kasparov (1996))**.** *A-T-menable groups satisfy the strong form of the Baum-Connes conjecture.*

1.3. **Examples.**

1.3.1. *Finite dimensional Hilbert spaces.* Let $\mathbb{E}^n$ be the n dimensional Euclidean space. An isometry of $\mathbb{E}^n$ either has a fixed point or it is a composition of a linear isometry and a translation by a vector fixed by the linear part.

Let λ be an isometry of $\mathbb{E}^n$ without a fixed point. Then the associated action of $\mathbb{Z}$ (by powers of λ) is proper (of course, if λ had a fixed vector the action would not be proper). Moreover there is the following result:

Theorem 1.9 (Bieberbach (1930))**.** *A finitely generated group with a proper isometric action on $\mathbb{E}^n$ is virtually Abelian.*

1.3.2. *A construction of an affine action.* Let (X, d) be a metric space with an action of G by isometries. Suppose we have

- a Hilbert space $\mathcal{H}$ with a unitary representation π of G,
- a continuous map $c : X \times X \to \mathcal{H}$ such that
  - $\forall\, x, y \in X\ g \in G\ \ c(gx, gy) = \pi(g)c(x, y)$,
  - $\forall\, x, y, z \in X\ \ c(x, y) + c(y, z) = c(x, z)$ (this is called the *Chasles' relation*),
  - there exists a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\left\|c(x, y)\right\|^2 = \varphi\big(d(x, y)\big)$ for all $x, y \in X$ (i.e. the norm of $c(x, y)$ depends only on $d(x, y)$).

Then to any $x_0 \in X$ we can associate an affine action α of G on $\mathcal{H}$ with linear part π such that $\left\|b(g)\right\|^2 = \varphi\big(d(gx_0, x_0)\big)$ for all $g \in G$. Indeed, we can put $b(g) = c(gx_0, x_0)$. By Chasles' relation $b \in Z^1(G, \pi)$.

If φ is a proper function (i.e. $\lim_{t\to\infty} \varphi(t) = +\infty$) and G acts properly on X then b is a proper cocycle and so G is a-T-menable.

Now let us give a concrete example of the situation described above. Let X be a tree: $X = (V, E)$. Let $\mathbb{E}$ be the set of oriented edges in X (each edge appears in $\mathbb{E}$ twice – with both orientations). Let $\mathcal{H} = \ell^2(\mathbb{E})$ and let π be the permutation representation (we assumed that G acts on X).

For any $x, y \in X$ (or more precisely $x, y \in V$) we have to define the vector $c(x, y) \in \ell^2(\mathbb{E})$. Let $e \in \mathbb{E}$ and let $[x, y]$ be the unique (oriented) geodesic path from x to y. We let

$$c(x,y)(e) = \begin{cases} 0 & \text{if } e \text{ is not in } [x,y] \\ +1 & \text{if } e \in [x,y] \text{ with the correct orientation} \\ -1 & \text{if } e \in [x,y] \text{ with the wrong orientation} \end{cases}$$

The Chasles' relation follows from the fact that all triangles in a tree are degenerate, so if x, y, z are vertices and we take a geodesic path from x to y and then from y to z then part of the path will have to be travelled in both directions and it will "cancel".

Moreover we have $\|c(x,y)\|^2 = 2d(x,y)$.

It follows that groups acting properly on a tree are a-T-menable (such groups are e.g. $\mathbb{F}_n$, $SL(\mathbb{Q}_p)$, etc.).

The above construction extends to groups acting on spaces with walls, CAT(0) cube complexes, spaces with measured walls,...

1.3.3. *Infinite dimensional Hilbert spaces.* In infinite dimensional Hilbert space we can have an "almost recurrent" isometry. We shall exhibit one on $\ell^2(\mathbb{N})$, where $\mathbb{N} = 1, 2, 3, \ldots$.

Let $\mathcal{F}(\mathbb{N}) = \mathbb{C}^{\mathbb{N}}$ (all functions $\mathbb{N} \to \mathbb{C}$). Define a linear operator on $\mathcal{F}(\mathbb{N})$ by

$$(Ua)_n = e^{\frac{2\pi i}{2^n}} a_n$$

for any $(a_n) \in \mathcal{F}(\mathbb{N})$. Note that U has no non zero fixed vector (because $0 \notin \mathbb{N}$).

Now let $w = (1, 1, \ldots) \in \mathcal{F}(\mathbb{N})$ and let $\alpha = T_w \circ U \circ T_W^{-1}$, where T_w is the translation by w. This means that

$$(\alpha(a))_n = e^{\frac{2\pi i}{2^n}} a_n + \left(1 - e^{\frac{2\pi i}{2^n}}\right)$$

for any $a = (a_n) \in \mathcal{F}(\mathbb{N})$.

The first claim is that $\alpha\big(\ell^2(\mathbb{N})\big) \subset \ell^2(\mathbb{N})$. Indeed, this is the case because the sequence (b_n) with $b_n = 1 - e^{\frac{2\pi i}{2^n}}$ belongs to $\ell^2(\mathbb{N})$.

Proposition 1.10 (Edelstein (1964))**.** *The map $\alpha\big|_{\ell^2(\mathbb{N})}$ is an isometry with unbounded orbits. Moreover there is a constant $R > 0$ such that*

$$\|\alpha^l(0)\| \le R$$

for infinitely many l's.

Proof. The only fixed point of U is $0 \in \mathcal{F}(\mathbb{N})$, so the only fixed point of α is w which does not belong to $\ell^2(\mathbb{N})$. Therefore α has no fixed point in $\ell^2(\mathbb{N})$. It follows that α has unbounded orbits (Proposition 1.5 and the dictionary above).

Now $\alpha^l(o) = w - U^l w$, so for $l = 2^k$ we have

$$\|\alpha^{2^k}(0)\|^2 = \sum_{n=1}^{\infty} \left|1 - e^{\frac{2\pi i 2^k}{2^n}}\right|^2 = \sum_{n=k+1}^{\infty} \left|1 - e^{\frac{2\pi i}{2^{(n-k)}}}\right|^2 = \sum_{t=1}^{\infty} \left|1 - e^{\frac{2\pi i}{2^t}}\right|^2.$$

We define R as the square root of the sum of the last series above. ☐

1.3.4. *Minimal actions.* An action is called *minimal* if it has dense orbits.

Question 1.11 (A. Navas)**.** *Which finitely generated groups admit an isometric minimal action on an infinite dimensional Hilbert space (separable for simplicity)?*

Proposition 1.12. *The wreath product $\mathbb{Z}^2 \wr \mathbb{Z}$ admits a minimal action on $\ell^2_{\mathbb{R}}(\mathbb{Z})$.*

Proof. First we identify $\mathbb{Z}^2$ with $\mathbb{Z}[\sqrt{2}]$. $\mathbb{Z}[\sqrt{2}]$ acts minimally on $\mathbb{R}$ by translation, so $\bigoplus_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ acts minimally by translations on $\ell^2_{\mathbb{R}}(\mathbb{Z})$ (because $\bigoplus_{\mathbb{Z}} \mathbb{R}$ is dense in $\ell^2_{\mathbb{R}}(\mathbb{Z})$). This action is equivariant with respect to the left regular representation of $\mathbb{Z}$, so it extends to an action of the wreath product. ☐

Theorem 1.13. *Every minimal isometric action of a finitely generated nilpotent group on a Hilbert space is an action by translations on a finite dimensional space.*

Let us conclude this section with an open question:

Question 1.14. *Can polycyclic groups act minimally isometrically on an infinite dimensional Hilbert space?*

2. Amenability & 1-cohomology

Definition 2.1. Let π be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$. We say that π *almost has invarian vectors* if

$$\forall\, \varepsilon > 0\ \forall\, K \Subset G\ \exists\, \xi \in \mathcal{H}\ \ \|\xi\| = 1,\ \sup_{g\in K} \big\| \pi(g)\xi - \xi \big\| < \varepsilon.$$

Example 2.2. As an example of the use of this notion let us state the following theorem:

Theorem 2.3 (Reiter's property (P_2))**.** *A locally compact group G is amenable if and only if the left regular representation λ_G on $L^2(G)$ almost has invariant vectors.*

Theorem 2.4 (Guichardet (1972))**.** *Let G be a σ-compact group and π a unitary representation of G with no non zero fixed vector. Then*

$$\begin{pmatrix} \text{π does not almost} \\ \text{have invariant vectors} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{The space $B^1(G,\pi)$ is closed} \\ \text{in the space $Z^1(G,\pi)$} \end{pmatrix}$$

Before proving this theorem let us state an immediate corollary.

Corollary 2.5. *If G is σ-compact and non compact then G is non amenable if and only if $B^1(G, \lambda_G)$ is closed in $Z^1(G, \lambda_G)$.*

In particular if G is amenable, σ-compact and non compact then we have $H^1(G, \lambda_G) \neq \{0\}$.

Proof of Theorem 2.4. Because G is σ-compact $Z^1(G,\pi)$ is a Frechet space. Consider the coboundary map $\partial : \mathcal{H}_\pi \to Z^1(G,\pi)$ (given by $\partial\xi(g) = \pi(g)\xi - \xi$).

We know that

- ∂ is linear,
- ∂ is continuous (because π is continuous),[b]
- ∂ is injective (because π has no non zero fixed vectors),
- the image of ∂ is, of course, $B^1(G,\pi)$.

We have the following chain of equivalences:

$$\Big(B^1(G,\pi) \text{ is closed in } Z^1(G,\pi) \Big)$$
$$\Updownarrow$$
$$\Big(\partial^{-1} \text{ is continuous} \Big)$$
$$\Updownarrow$$
$$\Big(\exists\, C > 0,\ K \Subset G\ \forall\, \xi \in \mathcal{H}_\pi\ \ \|\xi\| \le C \sup_{g\in K} \big\|\pi(g)\xi - \xi\big\| \Big)$$
$$\Updownarrow$$
$$\Big(\pi \text{ does not almost have invariant vectors} \Big)$$

The first equivalence follows from the closed graph theorem (the version for Frechet spaces, here we use σ-compactness of G). To see the second equivalence, recall the definition of the seminorms defining the topology of $Z^1(G,\pi)$. □

Exercise 2.6. Let $\mathbb{R}_d$ denote the the group of real numbers with discrete topology. Show that $\partial : \ell^2(\mathbb{R}) \to Z^1(G, \lambda_{\mathbb{R}_d})$ is a continuous isomorphism with discontinuous inverse (i.e. $H^1(G, \lambda_{\mathbb{R}_d}) = \{0\}$).

Why does it not contradict the closed graph theorem?

[b] Moreover ∂ has image in the subspace of bounded continuous maps $G \to \mathcal{H}$ and is continuous with the sup-norm on the latter space. Nevertheless the familiar version of the closed graph theorem for Banach spaces does not suffice to prove Theorem 2.4.

3. Property (BP_0)

Definition 3.1. A unitary representation π of a locally compact group G is called a C_0-*representation* or it is *mixing* if

$$\forall\, \xi, \eta \in \mathcal{H}_\pi \quad \lim_{g\to\infty} (\pi(g)\xi|\eta) = 0.$$

Example 3.2.

(1) Any representation of a compact group is C_0.
(2) The regular representation of any locally compact group is C_0.
(3) If G acts on a probability space $(X, \mathcal{B}, \mu)$ in a measure preserving way then we can consider the associated unitary representation π_X of G on $L^2_0(X,\mu)$, i.e. the orthogonal complement in $L^2(X,\mu)$ of the space of constant functions. We have

$$\Big(\pi_X \text{ is } \mathrm{C}_0\Big) \iff \Big(\text{The action of } G \text{ on } X \text{ is mixing}\Big).$$

(Recall that an action is mixing if for any $A, B \in \mathcal{B}$ we have $\lim_{g\to\infty} \mu(A\cap gB) = \mu(A)\mu(B)$.)

Definition 3.3. A locally compact group G has *property* (BP_0) if for every affine isometric action of G on a Hilbert space with C_0 linear part either the action has a fixed point or the action is metrically proper.

An equivalent definition is the following: G has property (BP_0) if and only if for any C_0-representation π and any $b \in Z^1(G,\pi)$ either b is bounded or b is proper (cf. Proposition 1.5). This explains the origin of the name of property (BP_0): "<u>B</u>ounded", "<u>P</u>roper" and "C<u>0</u>-representations".

Remark 3.4.

(1) Property (T) implies property (BP_0).
(2) The groups $SO(n,1)$ and $SU(n,1)$ have property (BP_0) and they do not have property (T) (Shalom, 2000).
(3) If H is a closed cocompact subgroup of G and H has property (BP_0) then G has (BP_0). This is because cocompactness of H in G guarantees that if restriction of $b \in Z^1(G,\pi)$ is bounded/proper then b must be bounded/proper.

Theorem 3.5 (de Cornulier, Tessera, Valette)**.** *Solvable groups have property* (BP_0).

Corollary 3.6. *Let G be a connected Lie group or a linear algebraic group over $\mathbb{Q}_p$ (or some other local field of characteristic* 0*). Then G has property* (BP_0)*.*

Proof. Such a group has a cocompact solvable subgroup. □

Theorem 3.5 is proved by induction on the solvability rank of G. The first step is provided by the following proposition:

Proposition 3.7. *Let G be a locally compact group with non compact center. Then G has* (BP_0)*.*

In particular every Abelian group has (BP_0). Proposition 3.7 has the following corollary:

Corollary 3.8 (Bekka, Cherix, Valette (1991))**.** *σ-compact, amenable groups are a-T-menable.*

Proof. If G is σ-compact and amenable then $H = G\times\mathbb{Z}$ is non compact, σ-compact and amenable. Therefore by the statement after Corollary 2.5 the group $H^1(H,\lambda_H)$ is not trivial. Take $b \in Z^1(H,\lambda_H) \setminus B^1(H,\lambda_H)$. By Proposition 3.7 the cocycle b is proper because it is not bounded, i.e. trivial (cf. Proposition 1.5). Thus b remains proper after restriction to G. Therefore G does admit a proper affine isometric action on a Hilbert space. □

Proof of Proposition 3.7. Let π be a C_0-representation of G and let $b \in Z^1(G,\pi)$. Assume that b is not proper. We must prove that b is bounded (cf. Proposition 1.5).
Claim: It is enough to show that $b\big|_{\mathcal{Z}(G)}$ is bounded.

Let us first prove that the above claim implies the proposition. Let α be the action associated to b: $\alpha(g)v = \pi(g)v + b(g)$. If $b\big|_{\mathcal{Z}(G)}$ is bounded then the fixed point set

$$\mathcal{H}^{\alpha\left(\mathcal{Z}(G)\right)}$$

is not empty. In fact this set consists of one point because if v_0, v_1 are fixed by $\alpha\big(\mathcal{Z}(G)\big)$ then $v_0 - v_1$ is fixed under $\pi\big(\mathcal{Z}(G)\big)$. Thus

$$\mathcal{Z}(G) \ni z \longmapsto (\pi(z)(v_0 - v_1)|v_0 - v_1)$$

is a constant C_0 function on a non compact group. It is therefore identically zero and consequently $v_0 = v_1$ (just evaluate this function at $1 \in \mathcal{Z}(G)$).

Moreover, since $\mathcal{Z}(G)$ is a normal subgroup of G, we have that $\mathcal{H}^{\alpha\left(\mathcal{Z}(G)\right)}$ is α-invariant. Therefore α has a globally fixed point and b is a coboundary. This proves the claim.

It remains to show that indeed $b\big|_{\mathcal{Z}(G)}$ is bounded. We assumed that b is not proper, so $\liminf_{g\to\infty}\big\|b(g)\big\| = C < +\infty$ in the sense that there is a net in G divergent to infinity (eventually outside of every compact set) for which the function $g \mapsto \big\|b(g)\big\|$ remains bounded. Now for any $z \in \mathcal{Z}(G)$ and $g \in G$ we have

$$\pi(g)b(z) + b(g) = b(gz) = b(zg) = \pi(z)b(g) + b(z)$$

(by the 1-cocycle relation), so that

$$b(z) = \big(1 - \pi(z)\big)b(g) + \pi(g)b(z). \tag{3.1}$$

Taking scalar product with $b(z)$ of both sides of (3.1) we obtain

$$(b(z)|b(z)) = \Big(\big(1 - \pi(z)\big)b(g)\Big|b(z)\Big) + (\pi(g)b(z)|b(z))$$

The absolute value of the first term on the right hand side is smaller than $2\big\|b(g)\big\|\big\|b(z)\big\|$ while the second term tends to 0 when $g \to \infty$. Taking g to infinity of G in such a way that $\big\|b(g)\big\|$ remains bounded we find that

$$\big\|b(z)\big\|^2 \leq 2C\big\|b(z)\big\|$$

and canceling $\big\|b(z)\big\|$ we obtain $\big\|b(z)\big\| \leq 2C$ for any $z \in \mathcal{Z}(G)$. □

4. Growth of cocycles

If G is a locally compact compactly generated group and S is a compact and symmetric generating set for G then we can define the word length function $|\cdot|_S$ on G by

$$|g|_S = \min\big\{n\big|\, g = s_1 s_2 \cdots s_n,\ s_i \in S\big\}$$

for any $g \in G$.

Now let π be a unitary representation of G and let $b \in Z^1(G, \pi)$. We arrive at the following question:

Question 4.1. *How fast does $\big\|b(g)\big\|$ grow with respect to $|g|_S$?*

Lemma 4.2. *We have $\big\|b(g)\big\| = O\big(|g|_S\big)$. More precisely*

$$\big\|b(g)\big\| \leq \Big(\max_{s\in S}\big\|b(s)\big\|\Big) \cdot |g|_S. \tag{4.1}$$

Proof. Let us first remark that whenever G acts by isometries on a metric space (X, d) and we chose an $x_0 \in X$ then

$$d(gx_0, x_0) \leq \Big(\max_{s\in S} d(sx_0, x_0)\Big) \cdot |g|_S. \tag{4.2}$$

Indeed, for $g = s_1 s_2 \cdots s_n$ with $n = |g|_S$ we have

$$\begin{aligned}
d(gx_0, x_0) &= d(s_1 s_2 \cdots s_n x_0, x_0) \\
&\le d(s_1 s_2 \cdots s_n x_0, s_1 s_2 \cdots s_{n-1} x_0) + d(s_1 s_2 \cdots s_{n-1} x_0, x_0) \\
&= d(s_n x_0, x_0) + d(s_1 s_2 \cdots s_{n-1} x_0, x_0) \\
&\le d(s_n x_0, x_0) + d(s_1 s_2 \cdots s_{n-1} x_0, s_1 s_2 \cdots s_{n-2} x_0) + d(s_1 s_2 \cdots s_{n-2} x_0, x_0) \\
&= d(s_n x_0, x_0) + d(s_{n-1} x_0, x_0) + d(s_1 s_2 \cdots s_{n-2} x_0, x_0) \\
&\;\;\vdots \\
&\le d(s_n x_0, x_0) + d(s_{n-1} x_0, x_0) + \cdots + d(s_1 x_0, x_0) \\
&\le n\Big(\max_{s \in S} d(sx_0, x_0)\Big)
\end{aligned}$$

because the action of G is isometric.

Now let α be the affine isometric action of G on $\mathcal{H}$ associated to b. Using (4.2) with $X = \mathcal{H}$ and $x_0 = 0$ we obtain precisely (4.1). □

Lemma 4.2 gives us one of the ways to see that $g \mapsto \big\|b(g)\big\|$ is a length function on G.

Lemma 4.3. *If $b \in \overline{B^1(G,\pi)}$ then $\big\|b(g)\big\| = o(|g|_S)$, i.e.*

$$\frac{\big\|b(g)\big\|}{|g|_S} \xrightarrow[|g|_S \to \infty]{} 0.$$

Proof. Let us fix $\varepsilon > 0$. There exists $b' \in B^1(G,\pi)$ such that

$$\max_{s \in S} \big\|b(s) - b'(s)\big\| < \tfrac{\varepsilon}{2}$$

(recall that $Z^1(G,\pi)$ has the topology of uniform convergence on compact sets). Therefore

$$\frac{\big\|b(g)\big\|}{|g|_S} \le \frac{\big\|b(g) - b'(g)\big\|}{|g|_S} + \frac{\big\|b'(g)\big\|}{|g|_S} \le \frac{\varepsilon}{2} + \frac{\big\|b'(g)\big\|}{|g|_S},$$

where in the last inequality we simply used (4.1) with b replaced by $b - b'$.

Now b' is a coboundary, so by Proposition 1.5 it is bounded and

$$\frac{\big\|b'(g)\big\|}{|g|_S} < \frac{\varepsilon}{2}$$

for sufficiently large $|g|_S$. □

4.1. **Application: a new look at an old proof.**

Theorem 4.4 (Von Neumann's mean ergodic theorem)**.** *Let U be a unitary operator on a Hilbert space $\mathcal{H}$. Then for any $v \in \mathcal{H}$ we have*

$$\tfrac{1}{n}\big(1 + U + U^2 + \cdots + U^{n-1}\big)v \xrightarrow[n \to \infty]{\|\cdot\|} Pv,$$

where P is the orthogonal projection onto $\ker(U - 1)$.

Proof. Let us define a unitary representation π of $\mathbb{Z}$ on $\mathcal{H}$ by $\pi(n) = U^n$. Also let $b \in Z^1(G,\pi)$ be the unique cocycle with $b(1) = v$. Using the cocycle relation (1.1) we find that

$$\begin{aligned}
b(n) = b\big((n-1)+1\big) &= U^{n-1}b(1) + b(n-1) \\
&= U^{n-1}b(1) + U^{n-2}b(1) + b(n-2) \\
&\;\;\vdots \\
&\;\; U^{n-1}b(1) + U^{n-2}b(1) + \cdots + Ub(1) + b(1) \\
&= \big(1 + U + U^2 + \cdots + U^{n-1}\big)v.
\end{aligned}$$

Let $\mathcal{H}_1 = P\mathcal{H}$ and $\mathcal{H}_0 = \mathcal{H}_1^\perp$. We have $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0$ and both subspaces are invariant for U. Let π_1 and π_0 be corresponding subrepresentations of π. Furthermore let

$$\begin{aligned} b_1(n) &= P\,b(n), \\ b_0(n) &= (1-P)b(n). \end{aligned}$$

Then $b_i \in Z^1(G,\pi_i)$ for $i = 1, 0$. On $\mathcal{H}_1$ the operator U acts as identity, so

$$b_1(n) = P(1+U+U^2+\dots+U^{n-1})v = \big(1+U+U^2+\dots+U^{n-1}\big)Pv = nPv.$$

Therefore $\frac{1}{n}b_1(n) = Pv$. On the other hand, we have

$$\mathcal{H}_0 = \ker(U-1)^\perp = \ker(U^*-1)^\perp = \overline{\operatorname{ran}(U-I)}$$

($U\xi = \xi$ if and only if $U^*\xi = \xi$). This means that $b_0(1) = (I-P)v$ is the limit of a sequence $(U-1)\xi_n$ for some $\xi_n \in \mathcal{H}$. It is easy to see that for each fixed k the vector $b_0(k)$ is the limit of $(\partial\xi_n)(k)$, so b_0 is in the closure of $B^1(G,\pi_0)$ in the topology of uniform convergence on compact subsets of $\mathbb{Z}$. By Lemma 4.3 we have

$$\frac{\big\|b_0(n)\big\|}{n} \xrightarrow[n\to\infty]{} 0.$$

□

The next exercise is a recap on Sections 1 and 4.

Exercise 4.5. Let α be an affine isometry of a Hilbert space $\mathcal{H}$. For any $\xi \in \mathcal{H}$ we have

$$\alpha(\xi) = U\xi + v$$

where U is a unitary operator and $v \in \mathcal{H}$ is a fixed vector. Let b be the cocycle on $\mathbb{Z}$ with $b(1) = v$. Prove that

(1) the following are equivalent:
   (a) α has a fixed point,
   (b) $v \in \operatorname{ran}(U-1)$,
   (c) b is bounded;
(2) the following are equivalent:
   (a) α almost has a fixed point, but no fixed point,
   (b) $v \in \overline{\operatorname{ran}(U-1)} \setminus \operatorname{ran}(U-1)$,
   (c) b is unbounded with $\big\|b(n)\big\| = o(n)$;
(3) the following are equivalent:
   (a) α does not almost have a fixed point,
   (b) $v \notin \overline{\operatorname{ran}(U-1)}$,
   (c) $\exists\, C > 0 \ \big\|b(n)\big\| \geq C|n|$.

Let us comment that part (3) of Exercise 4.5 is analogous to the finite dimensional situation of Subsubsection 1.3.1. The Edelstein example (Proposition 1.10) falls under case (2).

5. Applications to geometric group theory

Definition 5.1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a map.

(1) f is a *uniform embedding* if there exist functions $\rho_+, \rho_- : \mathbb{R}^+ \to \mathbb{R}$ such that $\lim\limits_{r\to+\infty} \rho_\pm(r) = +\infty$ and
$$\forall\, x_1, x_2 \in X \ \ \rho_-\big(d_X(x_1,x_2)\big) \leq d_Y\big(f((x_1), f(x_2)\big) \leq \rho_+\big(d_X(x_1,x_2)\big).$$
(2) f is a *quasi isometric embedding* if f is a uniform embedding for which the functions $\rho_\pm$ can be chosen to be affine functions.
(3) f is a *quasi isometry* if f is a quasi isometric embedding and there exists a quasi isometric embedding $g : Y \to X$ such that $f \circ g$ is a bounded distance from id_X and $g \circ f$ is a bounded distance from id_Y.

An example of a quasi isometry is the map $\mathbb{R} \ni x \mapsto [x] \in \mathbb{Z}$. Thus a quasi isometry need not be an isometry nor even a continuous map. Similarly a quasi isometric or uniform embedding need not be continuous nor an embedding.

Theorem 5.2 (Gromov (1979))**.** *For finitely generated groups being virtually nilpotent is an invariant of quasi isometries.*

Theorem 5.2 is really a restatement (possibly a weakening as well) of Gromov's theorem. What Gromov has in fact proved is that finitely generated virtually nilpotent groups are exactly the groups with polynomial growth.

Corollary 5.3 (Quasi isometric rigidity of $\mathbb{Z}^n$)**.** *If G is a finitely generated group quasi isometric to $\mathbb{Z}^n$ then G contains $\mathbb{Z}^n$ as a finite index subgroup.*

The next result is at first sight unrelated to previous statements, but we will see that it in fact is.

Theorem 5.4 (Bourgain (1984))**.** *The 3-regular tree does not embed quasi isometrically into a Hilbert space.*

Other results on quasi isometry invariants for finitely generated groups include:

Theorem 5.5. *(Erschler) Being virtually solvable is not a quasi isometry invariant property.*

Question 5.6. *Is being virtually polycyclic a quasi isometry invariant?*

Question 5.6 is open. It lead Yehuda Shalom to the following definition:

Definition 5.7. Let G be a locally compact group. We say that G belongs to the class $(\mathrm{AmenH_{FD}})$ if

(1) G is amenable,
(2) if a unitary representaition π of G satisfies $\overline{H^1}(G, \pi) \neq \{0\}$ then π contains a finite dimensional subrepresentation (cf. (1.2)).

The name of the class $(\mathrm{AmenH_{FD}})$ comes from "<u>Amen</u>ability", "co<u>H</u>omology" and "<u>F</u>inite <u>D</u>imension".

Theorem 5.8 (Shalom (2003))**.**

(1) *The following groups are in the class* $(\mathrm{AmenH_{FD}})$*:*
  - *connected solvable Lie groups,*
  - *virtually polycyclic groups,*
  - *semi direct products $\mathbb{Q}_p \rtimes \mathbb{Z}$ (where $\mathbb{Q}_p$ is the field of p-addic numbers and $\mathbb{Z}$ acts on its additive group by multiplication by powers of p),*
  - *lamplighter groups, i.e. groups of the form $F \wr \mathbb{Z}$, where F is a finite group.*

(2) *For finitely generated groups being in* $(\mathrm{AmenH_{FD}})$ *is a quasi isometry invariant.*
(3) *A finitely generated infinite group in* $(\mathrm{AmenH_{FD}})$ *admits a finite index subgroup which surjects onto $\mathbb{Z}$.*

Corollary 5.9 (of Theorem 5.8 (3))**.** *A group quasi isometric to a polycyclic group virtually surjects onto $\mathbb{Z}$.*

Question 5.10. *Which compactly generated groups admit a quasi isometric embedding into a Hilbert space?*

The group $\mathbb{Z}^n$ acts by translations on $\mathbb{E}^n$. The choice of any orbit gives a quasi isometric embedding of $\mathbb{Z}^n$ into $\mathbb{E}^n$. More generally any closed subgroup of $\mathrm{Isom}(\mathbb{E}^n)$ embeds quasi isometrically into $\mathbb{E}^n$. It is not easy to find other examples.

Remark 5.11. There are some negative results. For example the following:

Theorem 5.12 (Cheeger, Kleiner, Lee, Naor (2006))**.** *The discrete Heisenberg group does not embed quasi isometrically into ℓ^1.*

Of course ℓ^1 is not a Hilbert space, but we mention this result here because is solves (negatively) a conjecture coming from theoretical computer science.

Conjecture 5.13 (de Cornulier, Tessera, Valette)**.** *A compactly generated group which embeds quasi isometrically into a Hilbert space admits a proper isometric action on a finite dimensional Euclidean space. In particular, because of Bieberbach's theorem (Theorem 1.9), if G is finitely generated then it should be virtually nilpotent.*

We shall refer to Conjecture 5.13 as the CTV conjecture.

Remark 5.14.

(1) A non amenable finitely generated group cannot embed quasi isometrically into a Hilbert space. This is because of a deep result of Benjamini-Schramm (1998) which says that the Cayley graphs of such a group contains a quasi isometrically embedded copy of the 3-regular tree, and Bourgain's theorem (Theorem 5.4).
(2) A finitely generated solvable group which is not virtually nilpotent cannot be embedded quasi isometrically into a Hilbert space. The reason for this is a result of de Cornulier-Tessera (2006) that such a group contains a quasi isometrically embedded copy of the free semigroup on two generators.

Theorem 5.15 (de Cornulier, Tessera, Valette)**.** *The CTV conjecture holds for compactly generated groups in* (AmenH$_{\mathrm{FD}}$).

In particular we have

Corollary 5.16. *A virtually polycyclic group embeds quasi isometrically into a Hilbert space if and only if it is virtually Abelian.*

Compare this with the following result:

Theorem 5.17 (Pauls (2001))**.** *A virtually nilpotent group embeds quasi isometrically into a* CAT(0) *space if and only if it is virtually Abelian.*

The hypothesis of Theorem 5.17 is stronger than that of Corollary 5.16, but so is the thesis (Hilbert spaces are CAT(0), in fact they are prototypical examples of such spaces). The proofs are, however, very different.

Let us concentrate on another corollary of the CTV theorem.

Corollary 5.18 (quasi isometric rigidity of $\mathbb{Z}^n$)**.** *If G is a finitely generated group which is quasi isometric to $\mathbb{Z}^n$ then G has a finite index subgroup isomorphic to $\mathbb{Z}^n$.*

The proof of this result is independent of Gromov's theorem (Theorem 5.2).

Proof of Corollary 5.18. $\mathbb{Z}^n$ is in the class (AmenH$_{\mathrm{FD}}$). Therefore so is G by Theorem 5.8 (2). Also $\mathbb{Z}^n$ embeds quasi isometrically into a Hilbert space, thus so does G. By the CTV theorem G is virtually Abelian, so G has $\mathbb{Z}^m$ as a finite index subgroup. To see that $m = n$ we must consider growth which on one hand is a quasi isometry invariant and detects the rank of $\mathbb{Z}^k$. □

Remark 5.19. It is also possible to give an algebraic proof of Bourgain's theorem (Theorem 5.4) using the CTV theorem. The idea behind it is the following: it is known that there is an action of $SL_2(\mathbb{Q}_2)$ on the 3-regular tree T_3 (Serre's book "Trees"). Using this action one can show that T_3 is quasi isometric to $\mathbb{Q}_2 \rtimes \mathbb{Z}$. This last group is in (AmenH$_{\mathrm{FD}}$). Now all we need to do is show that $\mathbb{Q}_2 \rtimes \mathbb{Z}$ cannot act properly and isometrically on a finite dimensional Euclidean space.

Such an action would be a homomorphism $\mathbb{Q}_2 \rtimes \mathbb{Z} \to \mathrm{Isom}(\mathbb{E}^n)$ and by properness it would have a compact kernel. But the only compact normal subgroup of $\mathbb{Q}_2 \rtimes \mathbb{Z}$ is $\{1\}$, so $\mathbb{Q}_2 \rtimes \mathbb{Z}$ would have to embed into the Lie group $\mathrm{Isom}(\mathbb{E}^n)$. But Lie groups don't have small subgroups, and an embedding of $\mathbb{Q}_2 \rtimes \mathbb{Z}$ would contradict that.

5.1. Ideas on how to prove the CTV theorem.

Theorem 5.20 (Schönberg (1930))**.** *Let X be a set and let $\psi : X \times X \to \mathbb{R}^+$ be symmetric and equal 0 on the diagonal.*[c] *Further let $\mathcal{H}$ be a Hilbert space. Then there exists a map $f : X \to \mathcal{H}$ such that $\psi(x,y) = \|f(x) - f(y)\|^2$ if and only if ψ is conditionally negative definite, i.e. for any $n \in \mathbb{N}$, any $x_1, \dots, x_n \in X$ and any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^n \lambda_i = 0$ we have*

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \psi(x_i, x_j) \leq 0.$$

Moreover if a group G acts on X and ψ is G-invariant then f can be taken to be G-equivariant with respect to some isometric affine action of G on $\mathcal{H}$.

Lemma 5.21 (Gromov (for discrete groups))**.** *Let G be a compactly generated and amenable group. Let f be a uniform embedding of G into a Hilbert space $\mathcal{H}$ with control functions $\rho_\pm$. Then there exists a constant $A \geq 0$ (which can be taken $= 0$ if G is discrete) and an equivariant uniform embedding $\widetilde{f}$ of G into $\mathcal{H}$ with control functions $\rho_- - A$ and $\rho_+ + A$.*

Proof for G discrete. Set $\psi(x,y) = \|f(x) - x(y)\|^2$. We have

$$\rho_-\big(|x^{-1}y|_S\big)^2 \leq \psi(x,y) \leq \rho_+\big(|x^{-1}y|_S\big)^2. \tag{5.1}$$

Fix $x, y \in G$ and consider the function

$$u : G \ni g \longmapsto \psi(gx, gy).$$

u is bounded by the second inequality of (5.1). Let m be an invariant mean on $\ell^\infty(G)$ and define

$$\widetilde{\psi}(x,y) = m(u).$$

The function $\widetilde{\psi} : G \times G \to \mathbb{R}^+$ is then G-invariant and we have

$$\rho_-\big(|x^{-1}y|_S\big)^2 \leq \widetilde{\psi}(x,y) \leq \rho_+\big(|x^{-1}y|_S\big)^2.$$

All we need to see now is that $\widetilde{\psi}$ is conditionally negative definite and use Schönberg's theorem (Theorem 5.20).

For this we note that conditionally negative definite functions form a convex cone which is closed in the topology of pointwise convergence. Moreover the mean m is a weak limit of probability measures. □

References

[1] M. Bekka, P. de la Harpe, A. Valette: *Kazhdan's property* (T). Prebook available on the Geneva preprint server `http://www.unige.ch/math/`.

[2] Y. de Cornulier, R. Tessera, A. Valette: Three papers on `http://arXiv.org/`.

Solutions of exercises

Solution of Exercise 2.6. The regular representation of a non compact group does not have non zero fixed vectors, so ∂ is injective and continuous by the reasoning in the proof of Theorem 2.4. It remains to show that ∂ maps $\ell^2(\mathbb{R})$ onto $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$.

Let us skip ahead to the result that every Abelian group has property (BP_0) (it follows from Proposition 3.7). This means that $\mathbb{R}_d$ must have (BP_0). So if b is in $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$ then it must be either bounded (i.e. lie in $B^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$) or

$$\mathbb{R}_d \ni t \longmapsto \|b(t)\|$$

must be a proper function (preimage of a compact set is compact). Observe that existence of a proper continuous function on a locally compact space implies σ-compactness. Therefore there are no non zero proper cocycles (ones whose norm is a proper function). Therefore, by property (BP_0), there are no nontrivial cocycles in $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$. This means that ∂ maps $\ell^2(\mathbb{R})$ onto $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$.

[c] Such a ψ is then called a *symmetric kernel* with zero diagonal.

To see that ∂^{-1} is not continuous let us note that the topology of $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$ is the topology of pointwise convergence. Therefore if ∂^{-1} were continuous then for a net of cocycles (b_γ) with $b_\gamma(s) = \lambda_s\xi_\gamma - \xi_\gamma$ convergent at some point $t \in \mathbb{R} \setminus \{0\}$ the net (ξ_γ) would have to be convergent in $\ell^2(\mathbb{R})$. In other words the operator $(\lambda_t - I)^{-1}$ would extend from $\mathrm{ran}(\lambda_t - 1)$ to a continuous map of the Hilbert space $\ell^2(\mathbb{R})$. But for each $t \neq 0$ the spectrum of λ_t is the whole unit circle: it is non empty and $u_s\lambda_t u_s^* = e^{ist}\lambda_t$ for any $s \in \mathbb{R}$, where u_s is the unitary operator

$$(u_s\psi)(k) = e^{isk}\psi(k)$$

which shows that the spectrum is invariant under all rotations.

The reason why this does not contradict the closed graph theorem is that $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$ is not a Frechet space because uncountably many seminorms are needed to define its topology. ☐

Solution of Exercise 4.5. As in the proof of Theorem 4.4 the isometry α defines a representation π of $\mathbb{Z}$ by $\pi(n) = U^n$, where U is the linear part of α.

Now let us turn to the following observation: the map

$$\Psi : Z^1(\mathbb{Z}, \pi) \ni b \longmapsto b(1) \in \mathcal{H}$$

is an isomorphism of topological vector spaces. Indeed, any vector can be a value of a cocycle at the point $1 \in \mathbb{Z}$ and this value determines the cocycle uniquely (cf. proof of Theorem 4.4). This shows that Ψ is an isomorphism. Moreover the topology on $Z^1(\mathbb{Z}, \pi)$ is the topology of pointwise convergence (and value of a cocycle at any point $n \in Z$ is given by applying a fixed bounded operater to its value at $1 \in \mathbb{Z}$). This shows that Ψ is a homeomorphism.

It is easy to see that $\Psi\big(B^1(\mathbb{Z}, \pi)\big) = \mathrm{ran}(U - 1)$. Thus also $\Psi\big(\overline{B^1(\mathbb{Z}, \pi)}\big) = \overline{\mathrm{ran}(U - 1)}$.

Now recall the dictionary presented in Section 1 to see that we have the equivalences

$$(1a) \Longleftrightarrow (1b), \qquad (2a) \Longleftrightarrow (2b), \qquad (3a) \Longleftrightarrow (3b).$$

In order to have the whole exercise wrapped up we need one more remark, namely that if $v \notin \overline{\mathrm{ran}(U - 1)}$ then we have $Pv \neq 0$, where P is the projection onto $\ker(U - 1)$. Moreover by von Neumann's mean ergodic theorem we have

$$\tfrac{1}{n}b(n) \xrightarrow[n\to\infty]{} Pv,$$

so $\big\|b(v)\big\| \geq Cn$ for some constant $C > 0$ (e.g. $C = \frac{1}{2}\|Pv\|$).

Now we can finish the solution of our exercise. Equivalence between (1c) and (1a) is the content of Proposition 1.5.

From Lemma 4.3 we see that (3c) implies (3a) and (3b), and by the remark above (3b) implies (3c).

Finally by Proposition 1.5 and Lemma 4.3 we know that (2c) follows from (2a) and/or (2b). Conversely if (3c) is satisfied then b cannot be a coboundary (because it is unbounded), but $v = b(1)$ cannot at the same time lie outside $\overline{\mathrm{ran}(U - 1)}$ (again by the remark above). ☐